Lecture 14

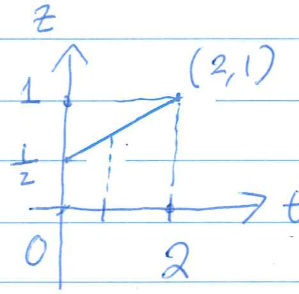
2020A
Fall, 2020.

Let's work out some examples.

e.g. Let $\gamma_1 : [0,1] \to \mathbb{R}^2$, $\gamma_1(t) = (t, 2t)$,

$\gamma_2 : [0,2] \to \mathbb{R}^2$, $\gamma_2(t) = (t+1, 2)$.

Write down a parametization of $\gamma = \gamma_1 + \gamma_2$ on $[0,1]$. Frist we have free to find the "break pt" of γ. Let it be $1/2$, we rescale γ_1 to $[0, 1/2] \to \mathbb{R}^2$ by setting it to be $t \mapsto (2t, 4t)$. Next, we want to transplant γ_2 from $[0,2]$ to $[1/2, 1]$: $t = 4z-2$ maps $[1/2, 1]$ to $[0,2]$



$\therefore \gamma_2$ is the same as $z \mapsto (4z-1, 2)$

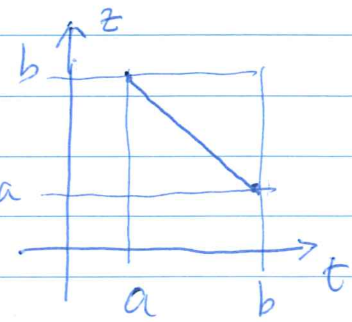
$z = \frac{1}{4}(t-2)+1$

$t = 4z-2$.

$[1/2, 1] \longrightarrow [0,2]$
$\quad z \qquad\qquad t$

$$\therefore \gamma(t) = \begin{cases} (2t, 4t), & t \in [0, \frac{1}{2}], \\ (4t-1, 2), & t \in [\frac{1}{2}, 1]. \end{cases}$$

e.g. Find $-\gamma$ if $\gamma : [a,b] \to \mathbb{R}^2$ or $\mathbb{R}^3$



Let $\gamma_1(t) = \gamma(a+b-t)$ $\quad t \in [a,b]$

As $\gamma_1(a) = \gamma(b)$, $\gamma_1(b) = \gamma(a)$, and

$$\gamma_1'(t) = -\gamma(a+b-t)$$

$$\Rightarrow |\gamma_1'(t)| = |\gamma(a+b-t)| > 0$$

$z = -(t-b)+a$, or
$t = b+a-z$

We see that as t runs from a to b, γ_1 runs from $\gamma(b)$ to $\gamma(a)$ monotonically, so $-\gamma = \gamma_1$ (or more precisely, $\gamma_1(t)$ is a parametrization for $-\gamma$.)

e.g. Let $\gamma(t) = (\cos t^2, \sin t^2)$, $t \in [0, \sqrt{2\pi}]$. Find its parametrization in arc-length.

$$\gamma'(t) = (-2t \sin t^2, 2t \cos t^2)$$

$$|\gamma'(t)| = 2t$$

$$s = \psi(t) = \int_0^t 2z\,dz = t^2$$

$$\therefore \varphi(s) = \psi^{-1}(s) = \sqrt{s} : [0, 2\pi] \to [0, \sqrt{2\pi}]$$

Let $\tilde{\gamma}(s) = \gamma(\varphi(s)) = \gamma(\sqrt{s}) = (\cos(\sqrt{s})^2, \sin(\sqrt{s})^2)$
$= (\cos s, \sin s)$, $s \in [0, 2\pi]$,

is the arc-length parametrization of γ.

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Line integral of the second kind is concerned with the integration of a vector field along a curve.

We introduce vector fields now.

The notion of a vector field comes up from physics long time ago. Now it is not only a fundamental notion in physics, but also important in mathematical analysis, dynamical systems and differential topology.

A vector field in a region in $\mathbb{R}^2$, $\mathbb{R}^3$, is simply

$$\vec{v} = (P(x,y), Q(x,y)), \quad \text{or} \quad (P(x,y,z), Q(x,y,z), R(x,y,z))$$

$$\text{or} \quad (L(x,y), M(x,y)), \quad \text{or} \quad (L(x,y,z), M(x,y,z), N(x,y,z))$$

where the components are functions in the region. A vector field is continuous (resp. C^1) if the components are continuous (resp. C^1).

Usually we represent the vector field by putting (P, Q) or (P, Q, R) at the base pt (x, y), or (x, y, z).

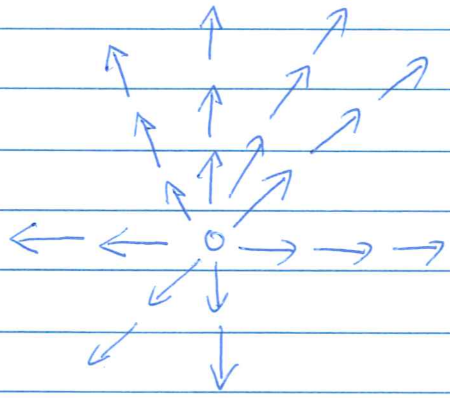
By the definition, a vector field is very general. However, in practise we better have some common vector fields in our mind. Let's look at some examples.

e.g. At each point (x, y) the vector field is a unit vector pointing out along the direction from the origin to (x, y).

$$\vec{v} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

$$= (\cos\theta, \sin\theta) \quad \text{(polar)}$$

at $(x, y) = (r\cos\theta, r\sin\theta)$.



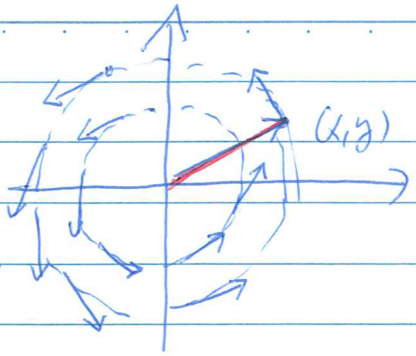
As $\vec{v}$ does not have a limit at $(0,0)$, this vector field is naturally defined in

$$\mathbb{R}^2 \setminus \{(0,0)\}.$$

e.g. $\vec{w} = (-y, x)$ at (x, y). $\vec{w}$ is a vector field defined in $\mathbb{R}^2$. From $(x, y)\cdot(-y, x) = 0$,

we see that it points to the orthogonal direction of (x, y). It is in the form of a rotation.

$|\vec{w}| = \sqrt{y^2+x^2} = r \to 0$ as $(x,y) \to 0$.



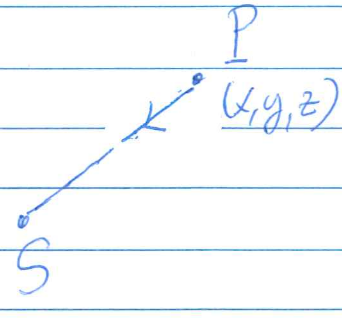
that's the strength of rotation $\to 0$
as $(x,y) \to 0$. So $\vec{w}$ is defined everywhere.
In fact, it is a C^1-vector field.

e.g. Let M be the mass of the sun located at $(0,0,0)$ and m be the mass of a planet. The force field acting on the planet is given by, its magnitude,

$$|\vec{F}| = \frac{GmM}{r^2}, \quad r - \text{distance from the sun}$$

and the direction is toward the sun, that is,

$$\frac{(x,y,z)}{r} \quad \text{(unit vector)}$$



$\therefore$

$$\vec{F} = -\frac{GmM}{r^3}(x,y,z) = -\frac{GmM}{(x^2+y^2+z^2)^{3/2}}(x,y,z).$$

$\vec{F}$ is C^1 in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, and $|\vec{F}(x,y,z)| \to \infty$ as $(x,y,z) \to (0,0,0)$,

e.g. As a good approximation, the gravity on Earth is a constant force field

$$\vec{G} = (0,0,-g), \quad g > 0.$$